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Complementarity of $\mathrm{su}_q(3)$ and $\mathrm{u}_q(2)$ and q-boson realization of the $\mathrm{su}_q(3)$ irreducible representations

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Abstract. The complementarity relationship (also termed duality) that arises between the irreps of the su(3) and u(2) Lie algebras when their direct sum $\mathrm{su}(3) \oplus \mathrm{u}(2)$ is embedded into a larger u(6) algebra and single-row irreps of the latter are considered is extended to the corresponding q-algebras $\mathrm{su}_q(3)$ and $\mathrm{u}_q(2)$. It is demonstrated by explicitly constructing the unique q-boson state that is simultaneously maximal in $\mathrm{su}_q(3)$ and $\mathrm{u}_q(2)$ for a given $\mathrm{u}_q(2)$ weight. In addition, the relations between the $\mathrm{su}_q(3)$ and $\mathrm{u}_q(2)$ Casimir operators resulting from their complementarity are explicitly found. Together with the q-Bargmann representation of q-boson operators, the complementarity relationship is then used to construct a Gel'fand–Tseitlin basis for arbitrary $\mathrm{su}_q(3)$ irreps in terms of q-boson operators.

1. Introduction

In recent years, a great deal of activity has been directed towards the exploration of quantized universal enveloping algebras, also called q-algebras or quantum groups (Jimbo 1985a, Drinfeld 1986). These new mathematical objects were developed in the theory of quantum integrable systems, where the Yang–Baxter equation plays a crucial role. Their relation to non-commutative geometry and the theory of knots and links has also attracted great interest. In physics, they have made their appearance in many fields, such as statistical mechanics, conformal field theory, quantum optics, molecular, atomic and nuclear spectroscopy (for reviews and references see e.g. Majid 1990, Zachos 1991).

In order to apply q-algebras in physics, one needs a well developed theory of their representations. Hopefully, the latter bears much similarity to that of ordinary Lie algebras. In particular, whenever q is not a root of unity, for any finite-dimensional irreducible representation (irrep) of a given simple Lie algebra, there is an irrep of the corresponding q-algebra that has the same dimension and the same weight spectrum, and so can be uniquely labelled by its highest weight (Lusztig 1988, Rosso 1988). For $\mathrm{u}_q(n)$, for instance, one can associate a unitary irrep with any n-row Young diagram.

The analogies between Lie algebras and q-algebras can also be extended to some subalgebra chains, such as $\mathrm{u}(n) \supset \mathrm{u}(n-1)$ and $\mathrm{u}_q(n) \supset \mathrm{u}_q(n-1)$, which admit the same branching rule (Jimbo 1985b, Ueno *et al* 1989). Both of these chains are

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canonical, which means that any irrep of $u(n-1)$ (resp. $u_q(n-1)$) is contained in any irrep of $u(n)$ (resp. $u_q(n)$) with a multiplicity at most equal to 1. Hence, as in the Lie algebraic case, the irreps of the q-subalgebras $u_q(n-1) \supset u_q(n-2) \supset \cdots \supset u_q(1)$ may serve to completely specify for any unitary irrep of $u_q(n)$ an orthonormal basis, the so-called Gel'fand–Tseitlin (GT) basis (Gel'fand and Tseitlin 1950). The latter may be constructed by means of lowering operators acting on its highest-weight vector (Ueno *et al* 1989, Quesne 1992).

The purpose of the present paper is to explore the extension to q-algebras of a special type of relationship, termed either complementarity (Moshinsky and Quesne 1970) or duality (Howe 1979), that arises between the irreps of some Lie algebras h_1, h_2 when their direct sum $h_1 \oplus h_2$ is embedded into a larger algebra g and some special irreps of the latter are considered. The prototype of this relationship is provided by the chain $u(6) \supset su(3) \oplus u(2)$: in the decomposition of the $u(6)$ single-row irreps into direct sums of $su(3)$ and $u(2)$ irreps, the latter appear in a one-to-one correspondence, associated irreps being characterized by the same two-row Young diagram (Moshinsky 1962, 1963, Baird and Biedenharn 1963). In the present work, the same branching rule will be shown to be valid for the corresponding q-algebra chain $u_q(6) \supset su_q(3) + u_q(2)$.†

To prove this result, it will be useful to realize the q-algebras $u_q(6)$, $su_q(3)$, and $u_q(2)$ in terms of the q-boson operators that were independently introduced by Biedenharn (1989) and Macfarlane (1989) to construct for $su_q(2)$ a q-analogue of the $su(2)$ Schwinger realization (Schwinger 1965). Similar realizations were also obtained for $su_q(n)$ (Sun and Fu 1989), and more generally for all classical q-algebras (Hayashi 1990). In the case of $su_q(n)$ with $n > 2$, they have been restricted so far to single-row irreps by using only n pairs of q-boson creation and annihilation operators. Since, in the present work, we shall deal with two-row irreps of $su_q(3)$, to get an appropriate realization we shall need 6 pairs of q-boson operators and make explicit use of the co-product of the $su_q(3)$ co-algebra structure.

Using q-boson realizations of q-algebras allows us to replace complicated q-commutation relations by q-differential calculus (Gasper and Rahman 1990). Such a simplification is based upon the q-analogue of the Bargmann representation (Bargmann 1961) of boson operators (Kulish and Damaskinsky 1990, Gray and Nelson 1990, Quesne 1991).

As a by-product of our demonstration procedure, we shall also obtain a concrete realization in terms of q-boson operators of the $su_q(3)$ GT basis, which was abstractly constructed by Ueno *et al* (1989) (see also Smirnov *et al* 1991). Together with a similar realization given for $u_q(2)$ by Biedenharn and Lohe (1991a, b), this paves the way for expressing the GT basis of two-row $u_q(n)$ irreps in terms of $2n$ pairs of q-boson operators.

The realization of the GT basis presented here differs from that based upon a recursive procedure, recently proposed for $su_q(2)$ by Quesne (1991), and for $su_q(3)$, and more generally $u_q(n)$, by Biedenharn and Lohe (1991a, b). According to the latter, $su_q(3)$ irrep GT basis vectors are obtained by $su_q(2)$ coupling a q-bosonic realization of an $su_q(2)$ irrep with an abstract vector in $u_q(2) \oplus u_q(1)$. This type of approach extends to q-algebras the $su(2)$ Dyson realization (Dyson 1956), generalized to other Lie algebras via vector coherent state theory (Deenen and Quesne 1984,

† After submitting the present paper, the author received a preprint of Smirnov and Tolstoy (1991), wherein a similar result is established.

Rowe 1984).

In the following section, we review the defining relations for $\mathrm{su}_q(n)$ and $\mathrm{u}_q(n)$ and apply them to $\mathrm{u}_q(6)$, $\mathrm{su}_q(3)$, and $\mathrm{u}_q(2)$. In section 3, we present q-boson realizations of the latter. In section 4, we demonstrate the complementarity between $\mathrm{su}_q(3)$ and $\mathrm{u}_q(2)$ by explicitly constructing the unique q-boson state that is simultaneously maximal in $\mathrm{su}_q(3)$ and $\mathrm{u}_q(2)$ for a given $\mathrm{u}_q(2)$ weight. In section 5, we find the expressions of the $\mathrm{su}_q(3)$ Casimir operators in terms of those of $\mathrm{u}_q(2)$, resulting from the complementarity of both q-algebras. Finally, in section 6, we construct a GT basis for the $\mathrm{su}_q(3)$ irreps in terms of q-boson operators.

2. Defining relations for $\mathrm{u}_q(6)$, $\mathrm{su}_q(3)$ and $\mathrm{u}_q(2)$

The $\mathrm{su}_q(n) \equiv \mathrm{U}_q(\mathrm{su}(n))$ q-algebra, corresponding to a one-parameter deformation of the universal enveloping algebra of $\mathrm{su}(n)$, is defined as the associative algebra over $\mathbb{C}$ generated by I, H_i, $X_i^{\pm}$, $i = 1, 2, \ldots, n-1$, and the commutation relations

$$[H_i, H_j] = 0 \tag{2.1a}$$

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \tag{2.1b}$$

$$[X_i^+, X_j^-] = \delta_{ij}\,[H_i] \equiv \delta_{ij}\,\frac{q^{H_i/2} - q^{-H_i/2}}{q^{1/2} - q^{-1/2}} \tag{2.1c}$$

together with the quadratic and cubic q-Serre relations given by

$$[X_i^{\pm}, X_j^{\pm}] = 0 \qquad j \neq i \pm 1 \qquad 1 \leqslant i, j \leqslant n-1 \tag{2.2}$$

and

$$(X_i^{\pm})^2 X_j^{\pm} - [2] X_i^{\pm} X_j^{\pm} X_i^{\pm} + X_j^{\pm} (X_i^{\pm})^2 = 0 \qquad j = i \pm 1 \quad 1 \leqslant i, j \leqslant n-1 \tag{2.3}$$

respectively (Jimbo 1985a). In (2.1b), a_{ij} is an element of the Cartan matrix associated with the classical simple Lie algebra A_{n-1}, i.e.

$$a_{ij} = \begin{cases} 2 & j = i \\ -1 & j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

In (2.3), [2] denotes a q-number, whose general definition is

$$[n] \equiv \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = q^{(n-1)/2} + q^{(n-3)/2} + \cdots + q^{-(n-1)/2} \qquad n \in \mathbb{Z}. \tag{2.5}$$

In (2.1c), this definition of q-numbers is extended to the commuting operators H_i.

Finally, the definition of the algebra is completed by assuming the Hermiticity properties

$$(H_i)^\dagger = H_i \qquad (X_i^{\pm})^\dagger = X_i^{\mp} \tag{2.6}$$

generalizing those of su(n). Such properties are consistent with (2.1)–(2.3) provided q is either a real number or a phase. Throughout this paper, we shall assume that $q \in \mathbb{R}^+$; the su(n) limit will correspond to $q \to 1$. The results presented here could be easily extended to the case where q is a phase different from a root of unity.

The q-algebra su$_q(n)$ has the structure of a Hopf algebra, admitting a co-product Δ, a co-unit ε and an antipode S, which are defined for the generators by

$$\Delta(H_i) = H_i \otimes I + I \otimes H_i \qquad \Delta\left(X_i^\pm\right) = X_i^\pm \otimes q^{H_i/4} + q^{-H_i/4} \otimes X_i^\pm \tag{2.7a}$$

$$\varepsilon(H_i) = \varepsilon\left(X_i^\pm\right) = 0 \tag{2.7b}$$

$$S(H_i) = -H_i \qquad S\left(X_i^\pm\right) = -q^{\pm 1/2} X_i^\pm . \tag{2.7c}$$

Both Δ and ε are algebra homomorphisms, whereas S is an anti-algebra and an anti-coalgebra map.

The q-algebra u$_q(n)$ is the algebra defined by the su$_q(n)$ generators plus an additional generator H_n commuting with all other generators. For H_n, the Hopf algebra and Hermiticity operations are the same as those for H_i, $i = 1, \ldots, n-1$.

The set of operators H_i, $X_i^\pm$ is the q-analogue of the Cartan–Chevalley basis of su(n), where the H_i are the generators of the Cartan subalgebra and the $X_i^\pm$ correspond to the roots $\pm\alpha_i$, α_i being the simple roots. However economical it may be, this choice of generators is not the most practical. It is indeed more interesting to use the Cartan–Weyl basis, whose n^2-1 generators will be denoted by $E_{ii} - E_{i+1,i+1}$, $1 \leqslant i \leqslant n-1$, and E_{ij}, $1 \leqslant i \neq j \leqslant n-1$.

The correspondence between these operators and the $3(n-1)$ Cartan–Chevalley generators is

$$H_i = E_{ii} - E_{i+1,i+1} \qquad X_i^+ = E_{i,i+1} \qquad X_i^- = E_{i+1,i} \qquad i = 1, \ldots, n-1 . \tag{2.8}$$

We define the additional generators of the Cartan–Weyl basis recursively by

$$E_{i,i+p} \equiv \left[E_{i,i+1}, E_{i+1,i+p}\right]_q \qquad E_{i+p,i} \equiv \left[E_{i+p,i+1}, E_{i+1,i}\right]_{q^{-1}} \qquad i = 1, \ldots, n-2 \qquad p = 2, \ldots, n-i \tag{2.9}$$

in terms of q-commutators of the type

$$[A, B]_{q^a} \equiv AB - q^{a/2} BA . \tag{2.10}$$

The cubic q-Serre relations (2.3) can now be expressed in the form of q-commutators as

$$\left[E_{i,i+1}, E_{i,i+2}\right]_{q^{-1}} = \left[E_{i+1,i+2}, E_{i,i+2}\right]_q = 0 \tag{2.11a}$$

$$\left[E_{i+2,i}, E_{i+1,i}\right]_q = \left[E_{i+2,i}, E_{i+2,i+1}\right]_{q^{-1}} = 0 . \tag{2.11b}$$

Equation (2.9) is not the only possible definition for the additional generators, and alternative definitions can actually be found in the literature. The Casimir operators to be considered in section 5 will be expressed in terms not only of the operators

(2.9), but also of operators differing from the latter by the substitution of q^{-1} for q. Both types of operators can be defined by

$$E^{(\pm)}_{i,i+p} \equiv \left[E_{i,i+1}, E^{(\pm)}_{i+1,i+p}\right]_{q^{\pm 1}} \qquad E^{(\pm)}_{i+p,i} \equiv \left[E^{(\pm)}_{i+p,i+1}, E_{i+1,i}\right]_{q^{\pm 1}}$$
$$i = 1, \ldots, n-2 \qquad p = 2, \ldots, n-i \tag{2.12}$$

so that the generators (2.9) reduce to

$$E_{i,i+p} = E^{(+)}_{i,i+p} \qquad E_{i+p,i} = E^{(-)}_{i+p,i}. \tag{2.13}$$

In the $u_q(n)$ case, the Cartan–Weyl basis, whose n^2 generators are E_{ij}, $i, j = 1, \ldots, n$, is obtained by setting

$$H_n = \sum_{i=1}^{n} E_{ii} \tag{2.14}$$

in addition to (2.8) and (2.9). The Hermiticity properties (2.6) then imply that

$$(E_{ij})^\dagger = E_{ji} \qquad j = i, i \pm 1 \tag{2.15a}$$

$$(E^{(\pm)}_{ij})^\dagger = E^{(\pm)}_{ji} \qquad j \neq i, i \pm 1. \tag{2.15b}$$

In the present paper, we shall deal with the $su_q(3)$ generators H_1, H_2, E_{ij}, $i \neq j$, $i, j = 1, 2, 3$. Since we shall also use $u_q(2)$ and $u_q(6)$, to distinguish their generators from those of $su_q(3)$ we shall denote them by $\mathcal{E}^{st}$, $s, t = 1, 2$, and $\mathbb{E}_{\mu\nu}$, $\mu, \nu = 1, \ldots, 6$, respectively. In the next section, we shall proceed to realize these three q-algebras in terms of q-boson operators.

3. q-boson realization of $u_q(6)$, $su_q(3)$ and $u_q(2)$

The q-algebra $w_q(m) \equiv U_q(w(m))$ (Hayashi 1990), which is the q-analogue of the Heisenberg–Weyl algebra $w(m)$, is defined as the associative algebra over $\mathbb{C}$ generated by I, N_μ, η_μ, ξ_μ, $\mu = 1, \ldots, m$, and the relations

$$[N_\mu, \eta_\nu] = \delta_{\mu\nu}\eta_\nu \qquad [N_\mu, \xi_\nu] = -\delta_{\mu\nu}\xi_\nu \tag{3.1a}$$

$$[\eta_\mu, \eta_\nu] = [\xi_\mu, \xi_\nu] = 0 \tag{3.1b}$$

$$[\xi_\mu, \eta_\nu] = 0 \qquad (\mu \neq \nu) \tag{3.1c}$$

$$[\xi_\mu, \eta_\mu]_q = q^{-N_\mu/2} \qquad [\xi_\mu, \eta_\mu]_{q^{-1}} = q^{N_\mu/2}. \tag{3.1d}$$

Instead of (3.1d), we may alternatively consider the relations

$$\xi_\mu\eta_\mu = [N_\mu + 1] \qquad \eta_\mu\xi_\mu = [N_\mu] \tag{3.2}$$

where the q-number notation of (2.5) has been used. We assume that the q-number operators N_μ are Hermitian, and that the q-creation operators η_μ are the Hermitian conjugates of the corresponding q-annihilation operators ξ_μ.

The operators N_μ, η_μ, ξ_μ act in a q-Fock space, spanned by the states

$$|\boldsymbol{n}\rangle = \prod_{\mu=1}^{m} \frac{(\eta_\mu)^{n_\mu}}{([n_\mu]!)^{1/2}} |0\rangle \tag{3.3}$$

where $|0\rangle$ is the q-boson vacuum state defined by

$$\xi_\mu|0\rangle = N_\mu|0\rangle = 0 \qquad \mu = 1, \ldots, m\,. \tag{3.4}$$

$[n_\mu]! \equiv [n_\mu]\,[n_\mu - 1] \ldots [1]$ is a q-factorial (with $[0]! \equiv 1$), and $\boldsymbol{n} = (n_1, \ldots, n_m)$ where $n_\mu \in \mathbb{N}$. The action of the operators on the basis states is given by

$$N_\mu|\boldsymbol{n}\rangle = n_\mu|\boldsymbol{n}\rangle \qquad \eta_\mu|\boldsymbol{n}\rangle = [n_\mu + 1]^{1/2}|\boldsymbol{n} + \boldsymbol{e}_\mu\rangle \qquad \xi_\mu|\boldsymbol{n}\rangle = [n_\mu]^{1/2}|\boldsymbol{n} - \boldsymbol{e}_\mu\rangle \tag{3.5}$$

where $\boldsymbol{e}_\mu$ is a row vector of dimension m with vanishing entries everywhere except for the μ component that has a value of unity.

In the q-Bargmann representation (Kulish and Damaskinsky 1990, Gray and Nelson 1990, Quesne 1991), the q-boson states and the q-boson operators are realized by

$$|\boldsymbol{n}\rangle \longrightarrow \prod_{\mu} \frac{(z_\mu)^{n_\mu}}{([n_\mu]!)^{1/2}} \qquad (z_\mu \in \mathbb{C}) \tag{3.6a}$$

$$\eta_\mu \longrightarrow z_\mu \tag{3.6b}$$

$$\xi_\mu \longrightarrow D_\mu \tag{3.6c}$$

$$N_\mu \longrightarrow z_\mu \partial_\mu \equiv z_\mu \partial/\partial z_\mu\,. \tag{3.6d}$$

The q-creation operators η_μ effect multiplication by the complex variables z_μ, while the q-number operators N_μ measure the degree in z_μ as in the Bargmann representation of boson operators (Bargmann 1961). However the q-annihilation operators ξ_μ are now represented by the q-differentiation operators $D_\mu \equiv D_{z_\mu}$, defined by

$$D_z f(z) = \frac{f(q^{1/2}z) - f(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})z}\,. \tag{3.7}$$

Some properties of these finite-difference operators to be used in the following sections have been reviewed by Gray and Nelson (1990).

The Cartan–Chevalley generators of $\mathrm{u}_q(n)$ can be realized in terms of the generators of $w_q(m)$ with $m = n$ (Hayashi 1990). In particular, for $\mathrm{u}_q(6)$ we obtain

$$\mathbb{E}_{\mu\mu} = N_\mu \qquad \mu = 1, \ldots, 6 \tag{3.8a}$$

$$\mathbb{E}_{\mu,\mu+1} = \eta_\mu \xi_{\mu+1} \qquad \mathbb{E}_{\mu+1,\mu} = \eta_{\mu+1}\xi_\mu \qquad \mu = 1, \ldots, 5\,. \tag{3.8b}$$

By using (2.9) and (3.1), the remaining generators of the $u_q(6)$ Cartan–Weyl basis can be expressed as

$$\mathbb{E}_{\mu,\mu+p} = q^{-(N_{\mu+1}+\cdots+N_{\mu+p-1})/2}\eta_\mu\xi_{\mu+p} \qquad \mathbb{E}_{\mu+p,\mu} = q^{(N_{\mu+1}+\cdots+N_{\mu+p-1})/2}\eta_{\mu+p}\xi_\mu$$
$$\mu = 1,\ldots,4 \qquad p = 2,\ldots,6-\mu\,. \tag{3.9}$$

In this realization, the $u_q(6)$ unitary irreps are characterized by a single-row Young diagram $[h0\ldots0]$, where $h \in \mathbb{N}$, and their highest-weight state $|h\rangle$ satisfies the equations

$$\mathbb{E}_{\mu\mu}|h\rangle = \delta_{\mu,1}|h\rangle \qquad \mu = 1,\ldots,6 \tag{3.10a}$$

$$\mathbb{E}_{\mu,\mu+1}|h\rangle = 0 \qquad \mu = 1,\ldots,5\,. \tag{3.10b}$$

To obtain realizations of $su_q(3)$ and $u_q(2)$, let us now replace the single index μ by a pair of indices i, s, where i and s run over 1, 2, 3, and 1, 2, respectively, and their values are enumerated in lexical order, i.e.

$$\mu \to is: \qquad 1\to 11 \quad 2\to 12 \quad 3\to 21 \quad 4\to 22 \quad 5\to 31 \quad 6\to 32\,. \tag{3.11}$$

Then $\mathbb{E}_{\mu\nu}$, N_μ, η_μ, ξ_μ, z_μ, ∂_μ, and D_μ become $\mathbb{E}_{ij}^{st}$, N_i^s, η_i^s, ξ_i^s, z_i^s, ∂_i^s, and D_i^s, respectively.

For each s value, we can obtain a realization of $su_q(3)$ in terms of N_i^s, η_i^s, ξ_i^s, $i = 1, 2, 3$, of the same type as that given above for $u_q(6)$. By using the co-product (2.7*a*), the two independent realizations of $su_q(3)$, corresponding to $s = 1$ and $s = 2$ respectively, can be combined to give the following $su_q(3)$ Cartan–Chevalley generators:

$$H_i = E_{ii} - E_{i+1,i+1} = N_i^1 + N_i^2 - N_{i+1}^1 - N_{i+1}^2 \tag{3.12a}$$

$$E_{i,i+1} = \eta_i^1\xi_{i+1}^1 q^{(N_i^2-N_{i+1}^2)/4} + q^{-(N_i^1-N_{i+1}^1)/4}\eta_i^2\xi_{i+1}^2 \tag{3.12b}$$

$$E_{i+1,i} = \eta_{i+1}^1\xi_i^1 q^{(N_i^2-N_{i+1}^2)/4} + q^{-(N_i^1-N_{i+1}^1)/4}\eta_{i+1}^2\xi_i^2 \tag{3.12c}$$

where, for simplicity's sake, we have dropped the tensor product symbol $\otimes$. By using (2.9) and (3.1), the two additional generators of the Cartan–Weyl basis can be expressed as

$$E_{13} = E_{13}^{(+)} = q^{-N_2^1/2}\eta_1^1\xi_3^1 q^{(N_1^2-N_3^2)/4} + q^{-(N_1^1-N_3^1)/4}q^{-N_2^2/2}\eta_1^2\xi_3^2$$
$$+ (q^{-1/4} - q^{3/4})q^{-(N_2^1-N_3^1)/4}\eta_1^1\xi_2^1 q^{(N_1^2-N_2^2)/4}\eta_2^2\xi_3^2 \tag{3.13a}$$

$$E_{31} = E_{31}^{(-)} = q^{N_2^1/2}\eta_3^1\xi_1^1 q^{(N_1^2-N_3^2)/4} + q^{-(N_1^1-N_3^1)/4}q^{N_2^2/2}\eta_3^2\xi_1^2$$
$$+ (q^{1/4} - q^{-3/4})q^{-(N_1^1-N_2^1)/4}\eta_3^1\xi_2^1 q^{(N_2^2-N_3^2)/4}\eta_2^2\xi_1^2\,. \tag{3.13b}$$

Note that, for $i \neq j$, one simply goes from E_{ij} to E_{ji} by permuting the indices is with js for $s = 1$, 2, and by changing q into q^{-1}. A similar calculation shows that $E_{13}^{(-)}$ and $E_{31}^{(+)}$ can be obtained from $E_{13}^{(+)}$ and $E_{31}^{(-)}$, respectively, by permuting the indices $i1$ with $i2$ for $i = 1, 2, 3$, and by changing q into q^{-1}.

In the same way, for each i value, we can define a realization of $u_q(2)$ in terms of N_i^s, η_i^s, ξ_i^s, $s = 1, 2$, then combine the three independent realizations so obtained by using the co-product (2.7*a*) to get the following $u_q(2)$ generators:

$$\mathcal{E}^{ss} = N_1^s + N_2^s + N_3^s \tag{3.14a}$$

$$\mathcal{E}^{st} = \eta_1^s \xi_1^t q^{(N_2^1 - N_2^2 + N_3^1 - N_3^2)/4} + q^{-(N_1^1 - N_1^2)/4} \eta_2^s \xi_2^t q^{(N_3^1 - N_3^2)/4} + q^{-(N_1^1 - N_1^2 + N_2^1 - N_2^2)/4} \eta_3^s \xi_3^t \qquad (s \neq t). \tag{3.14b}$$

It is easily shown that:

Lemma 1. The $su_q(3)$ and $u_q(2)$ q-algebras, whose Cartan–Chevalley generators are given in (3.12) and (3.14) respectively, are subalgebras of the $u_q(6)$ q-algebra defined in (3.8).

Proof. The operators (3.12) and (3.14) can be expressed in terms of the Cartan–Weyl generators of $u_q(6)$ as

$$\begin{aligned}
H_i &= \mathbb{E}_{ii}^{11} + \mathbb{E}_{ii}^{22} - \mathbb{E}_{i+1,i+1}^{11} - \mathbb{E}_{i+1,i+1}^{22} \\
E_{i,i+1} &= \mathbb{E}_{i,i+1}^{11} q^{(3\mathbb{E}_{ii}^{22} - \mathbb{E}_{i+1,i+1}^{22})/4} + q^{-(\mathbb{E}_{ii}^{11} - 3\mathbb{E}_{i+1,i+1}^{11})/4} \mathbb{E}_{i,i+1}^{22} \\
E_{i+1,i} &= \mathbb{E}_{i+1,i}^{11} q^{-(\mathbb{E}_{ii}^{22} + \mathbb{E}_{i+1,i+1}^{22})/4} + q^{-(\mathbb{E}_{ii}^{11} + \mathbb{E}_{i+1,i+1}^{11})/4} \mathbb{E}_{i+1,i}^{22}
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\mathcal{E}^{ss} &= \mathbb{E}_{11}^{ss} + \mathbb{E}_{22}^{ss} + \mathbb{E}_{33}^{ss} \\
\mathcal{E}^{st} &= \mathbb{E}_{11}^{st} q^{(\mathbb{E}_{22}^{11} - \mathbb{E}_{22}^{22} + \mathbb{E}_{33}^{11} - \mathbb{E}_{33}^{22})/4} + q^{-(\mathbb{E}_{11}^{11} - \mathbb{E}_{11}^{22})/4} \mathbb{E}_{22}^{st} q^{(\mathbb{E}_{33}^{11} - \mathbb{E}_{33}^{22})/4} \\
&\quad + q^{-(\mathbb{E}_{11}^{11} - \mathbb{E}_{11}^{22} + \mathbb{E}_{22}^{11} - \mathbb{E}_{22}^{22})/4} \mathbb{E}_{33}^{st} \qquad (s \neq t).
\end{aligned} \tag{3.16}$$

□

In the limit $q \to 1$, the operators (3.15) and (3.16) become su(3) and u(2) generators, given by

$$H_i = \sum_s (\mathbb{E}_{ii}^{ss} - \mathbb{E}_{i+1,i+1}^{ss}) \qquad E_{ij} = \sum_s \mathbb{E}_{ij}^{ss} \qquad (i \neq j) \tag{3.17}$$

and

$$\mathcal{E}^{st} = \sum_i \mathbb{E}_{ii}^{st} \tag{3.18}$$

respectively. Since the latter commute with one another, they provide a realization of the direct sum of Lie algebras su(3) ⊕ u(2), embedded into the larger algebra u(6) generated by $\mathbb{E}_{ij}^{st}$.

To determine what is the q-analogue of this property, it is useful to establish the following result:

Lemma 2. The Cartan–Chevalley generators (3.12) and (3.14) of $su_q(3)$ and $u_q(2)$ commute with one another.

Proof. Consider for instance the commutator of $\mathcal{E}^{12}$ and E_{12}. From (3.15) and (3.16), it can be written as

$$\begin{aligned}
[\mathcal{E}^{12}, E_{12}] &= q^{(3\mathbb{E}_{11}^{22}+\mathbb{E}_{22}^{11}-2\mathbb{E}_{22}^{22}+\mathbb{E}_{33}^{11}-\mathbb{E}_{33}^{22}+3)/4}\left[\mathbb{E}_{11}^{12}, \mathbb{E}_{12}^{11}\right]_{q^{-1}} \\
&+ q^{-(\mathbb{E}_{11}^{11}-4\mathbb{E}_{22}^{11}+\mathbb{E}_{22}^{22}-\mathbb{E}_{33}^{11}+\mathbb{E}_{33}^{22}-1)/4}\left[\mathbb{E}_{11}^{12}, \mathbb{E}_{12}^{22}\right]_{q^{-1}} \\
&+ q^{-(\mathbb{E}_{11}^{11}-4\mathbb{E}_{11}^{22}+\mathbb{E}_{22}^{22}-\mathbb{E}_{33}^{11}+\mathbb{E}_{33}^{22}+1)/4}\left[\mathbb{E}_{22}^{12}, \mathbb{E}_{12}^{11}\right]_{q} \\
&+ q^{-(2\mathbb{E}_{11}^{11}-\mathbb{E}_{11}^{22}-3\mathbb{E}_{22}^{11}-\mathbb{E}_{33}^{11}+\mathbb{E}_{33}^{22}+3)/4}\left[\mathbb{E}_{22}^{12}, \mathbb{E}_{12}^{22}\right]_{q} \\
&+ q^{-(\mathbb{E}_{11}^{11}-4\mathbb{E}_{11}^{22}+\mathbb{E}_{22}^{11})/4}\left[\mathbb{E}_{33}^{12}, \mathbb{E}_{12}^{11}\right] \\
&+ q^{-(2\mathbb{E}_{11}^{11}-\mathbb{E}_{11}^{22}-2\mathbb{E}_{22}^{11}-\mathbb{E}_{22}^{22})/4}\left[\mathbb{E}_{33}^{12}, \mathbb{E}_{12}^{22}\right] .
\end{aligned} \tag{3.19}$$

On the right-hand side of (3.19), the first and fourth q-commutators vanish due to the cubic q-Serre relations (2.11a) and the correspondence (3.11); the same is true for the two commutators as a consequence of the quadratic q-Serre relations (2.2) and the definition (2.9). From (3.1), (3.8) and (3.9), the second and third q-commutators become

$$\left[\mathbb{E}_{11}^{12}, \mathbb{E}_{12}^{22}\right]_{q^{-1}} = \left[\eta_1^1\xi_1^2, q^{-N_2^1/2}\eta_1^2\xi_2^2\right]_{q^{-1}} = q^{(N_1^2-N_2^1)/2}\eta_1^1\xi_2^2 = q^{\mathbb{E}_{11}^{22}}\mathbb{E}_{12}^{12} \tag{3.20a}$$

$$\left[\mathbb{E}_{22}^{12}, \mathbb{E}_{12}^{11}\right]_{q} = \left[\eta_2^1\xi_2^2, q^{-N_1^2/2}\eta_1^1\xi_2^1\right]_{q} = -q^{-(N_1^2-N_2^1-1)/2}\eta_1^1\xi_2^2 = -q^{(\mathbb{E}_{22}^{11}+1/2)}\mathbb{E}_{12}^{12} . \tag{3.20b}$$

Hence the right-hand side of (3.19) is equal to zero.

The vanishing of the remaining commutators can be proved in the same way. □

Remark. The operators (3.15) and (3.16) do not commute with one another for any realization of $u_q(6)$. The demonstration of Lemma 2 indeed depends in an essential way on the q-boson realization (3.8), since the relation between $\left[\mathbb{E}_{11}^{12}, \mathbb{E}_{12}^{22}\right]_{q^{-1}}$ and $\left[\mathbb{E}_{22}^{12}, \mathbb{E}_{12}^{11}\right]_{q}$ that follows from (3.20) cannot be derived from the $u_q(6)$ defining equations with the help of the q-Jacobi identity. This result contrasts with the $q \to 1$ limit, wherein the commuting property of the su(3) and u(2) generators (3.17) and (3.18) is independent of the realization chosen for the u(6) generators.

Since in the q-algebra case, we cannot consider a direct sum of algebras because $su_q(3)$ and $u_q(2)$ have the unit operator in common, let us make the following definition:

Definition. Let $su_q(3) + u_q(2) = U_q(su(3) \oplus u(2))$ denote the associative algebra over $\mathbb{C}$ generated by I, H_i, $E_{i,i+1}$, $E_{i+1,i}$, $\mathcal{E}^{st}$, i, s, t = 1, 2, where H_i, $E_{i,i+1}$, $E_{i+1,i}$, and $\mathcal{E}^{st}$ satisfy the defining relations of $su_q(3)$ and $u_q(2)$ respectively, and commute with one another.

The results obtained in the present section can be summed up in the form of a theorem.

Theorem 1. The operators (3.8), (3.12) and (3.14) provide a realization of the q-algebra chain

$$u_q(6) \supset su_q(3) + u_q(2). \tag{3.21}$$

4. Complementarity of $\mathrm{su}_q(3)$ and $\mathrm{u}_q(2)$

In the present section, we consider the polynomials in the q-creation operators $P(\eta_i^s)$ that satisfy the equations

$$\mathcal{E}^{11}P(\eta_i^s)|0\rangle = h_1 P(\eta_i^s)|0\rangle \qquad \mathcal{E}^{22}P(\eta_i^s)|0\rangle = h_2 P(\eta_i^s)|0\rangle \tag{4.1a}$$

$$\mathcal{E}^{12}P(\eta_i^s)|0\rangle = 0 \tag{4.1b}$$

and

$$H_1 P(\eta_i^s)|0\rangle = k_1 P(\eta_i^s)|0\rangle \qquad H_2 P(\eta_i^s)|0\rangle = k_2 P(\eta_i^s)|0\rangle \tag{4.2a}$$

$$E_{12}P(\eta_i^s)|0\rangle = E_{23}P(\eta_i^s)|0\rangle = 0 \tag{4.2b}$$

where h_1, h_2, k_1, $k_2 \in \mathbb{N}$ and $h_1 \geqslant h_2$. The corresponding q-boson states $P(\eta_i^s)|0\rangle$ are then simultaneously highest-weight states of $\mathrm{u}_q(2)$ and $\mathrm{su}_q(3)$ irreps characterized by Young patterns $[h_1 h_2]$ and $[k_1 + k_2, k_2, 0] \equiv (k_1 k_2)$. To keep the notations as simple as possible, we have dropped the indices h_1, h_2, k_1, k_2, α that should be appended to the symbol P in (4.1) and (4.2) (with α distinguishing between independent solutions corresponding to the same values of h_1, h_2, k_1, k_2).

We start by solving the system of equations (4.1). The results can be stated in the form of a lemma. In the latter, use is made of q-binomial coefficients, defined by

$$\begin{bmatrix} m \\ k \end{bmatrix} \equiv \begin{cases} \dfrac{[m]!}{[k]!\,[m-k]!} & 0 \leqslant k \leqslant m \\ 0 & k < 0 \quad \text{or} \quad k > m\,. \end{cases} \tag{4.3}$$

Lemma 3. The simultaneous solutions of (4.1) can be written as

$$P(\eta_i^s)|0\rangle = \sum_{abcd} A_{cd}^{(ab)} (\eta_1^1)^{h_1-c} (\eta_2^1)^{c-d} (\eta_3^1)^d (\eta_1^2)^{h_2-a-b+c} (\eta_2^2)^{a-c+d} (\eta_3^2)^{b-d} |0\rangle \tag{4.4}$$

where the summations over a, b, c, d run over those non-negative integers such that $h_2 \leqslant a+b \leqslant h_1+h_2$, $a+b-h_2 \leqslant c \leqslant \min(a+b, h_1)$, $\max(0, c-a) \leqslant d \leqslant \min(b, c)$. For given (ab), all the coefficients $A_{cd}^{(ab)}$ corresponding to different c, d values can be expressed in terms of those associated with the minimal c value as follows:

$$A_{cd}^{(ab)} = (-1)^{h_2-a-b+c} q^{(-(h_1-h_2+b+2)(h_2-a-b+c)+(2h_2-a-b)d)/4} \times \sum_{t=\max(0,b-h_2)}^{\min(b,a+b-h_2)} q^{(2c-a-b)t/4} \begin{bmatrix} b-t \\ b-d \end{bmatrix} \begin{bmatrix} h_2-b+t \\ a-c+d \end{bmatrix} A_{a+b-h_2,t}^{(ab)}\,. \tag{4.5}$$

For $h_2 \leqslant a+b \leqslant h_1$, the latter are linearly independent, whereas for $h_1 < a+b \leqslant h_1+h_2$ they satisfy the relations

$$\sum_{t=\max(0,b-h_2)}^{\min(b,a+b-h_2)} q^{(2h_1-a-b+2)t/4} \begin{bmatrix} b-t \\ b-d \end{bmatrix} \begin{bmatrix} h_2-b+t \\ a-h_1+d-1 \end{bmatrix} A_{a+b-h_2,t}^{(ab)} = 0 \tag{4.6}$$

where d runs over all the integers such that $\max(0, h_1-a+1) \leqslant d \leqslant \min(b, h_1+1)$.

Proof. In this proof as in the following ones, it is advantageous to use the q-Bargmann representation of q-boson operators so that all operations reduce to q-differentiations.

From (3.14*a*) and (3.6*d*), it is obvious that the simultaneous solutions of the system (4.1*a*) can be written in the form (4.4), where the coefficients $A_{cd}^{(ab)}$ are defined as vanishing whenever the inequalities $a \geqslant 0$, $b \geqslant 0$, $a+b \leqslant h_1+h_2$, $\max(0, a+b-h_2) \leqslant c \leqslant \min(h_1, a+b)$, $\max(0, c-a) \leqslant d \leqslant \min(b, c)$ are not fulfilled, but are otherwise arbitrary.

Introducing now (4.4) into (4.1*b*) and taking (3.14*b*) and (3.6*b*, *c*, *d*) into account lead to the following recursion relation for the coefficients $A_{cd}^{(ab)}$ with fixed (ab) satisfying the conditions $a \geqslant 0$, $b \geqslant 0$, $a+b \leqslant h_1+h_2$:

$$q^{(2c-a-b)/4}[h_2-a-b+c]A_{cd}^{(ab)} + q^{-(h_1-h_2+a+2b-2c-2d+2)/4}[a-c+d+1]A_{c-1,d}^{(ab)} + q^{-(h_1-h_2+b-2d+2)/4}[b-d+1]A_{c-1,d-1}^{(ab)} = 0. \tag{4.7}$$

In (4.7), c varies in the range $\max(0, a+b-h_2+1) \leqslant c \leqslant \min(h_1+1, a+b)$. Hence we have to distinguish the following cases:

$$\begin{array}{lll} \text{I.} & 0 \leqslant a+b < h_2 & 0 \leqslant c \leqslant a+b \\ \text{II.} & h_2 \leqslant a+b \leqslant h_1 & a+b-h_2+1 \leqslant c \leqslant a+b \\ \text{III.} & h_1 < a+b \leqslant h_1+h_2 & a+b-h_2+1 \leqslant c \leqslant h_1+1 \end{array} \tag{4.8}$$

together with $\max(0, c-a) \leqslant d \leqslant \min(b, c)$.

In case I, for $c = d = 0$ equation (4.7) gives $A_{00}^{(ab)} = 0$. Then by induction over c, one obtains $A_{cd}^{(ab)} = 0$ for any allowed values of c and d.

In case II, the solution of (4.7) is given by (4.5), where the coefficients $A_{a+b-h_2,t}^{(ab)}$ remain undetermined because there is no equation (4.7) for $c = a+b-h_2$. Equation (4.5) indeed reduces to an identity for $c = a+b-h_2$ and can be proved by induction over c in the range $a+b-h_2+1 \leqslant c \leqslant a+b$ by using the definition (4.3) of q-binomial coefficients and some elementary properties of q-numbers.

The treatment of case III is similar to that of case II for $a+b-h_2+1 \leqslant c \leqslant h_1$, but there is now an extra condition coming from (4.7) for $c = h_1+1$. It amounts to setting the right-hand side of (4.5) equal to zero for this c value, thereby leading to (4.6). □

We now take advantage of lemma 3 to find the simultaneous solutions of (4.1) and (4.2). In the following lemma, we make use of the q-binomial formula (Gasper and Rahman 1990)

$$(x; y)_q^m \equiv \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} x^{m-k} y^k \qquad m \in \mathbb{N}. \tag{4.9}$$

The q-binomial satisfies the following properties:

$$(x; y)_q^0 = 1 \tag{4.10}$$

$$(x;y)_q^m = \prod_{k=0}^{m-1} \left(x + q^{k-(m-1)/2} y\right) \qquad m \in \mathbb{N}^+ \tag{4.11}$$

$$(x;y)_q^m = \left(x; q^{\pm n/2} y\right)_q^{m-n} \left(x; q^{\mp(m-n)/2} y\right)_q^n \qquad n = 0, 1, \ldots, m \tag{4.12}$$

$$D_x (x;y)_q^m = [m] \, (x;y)_q^{m-1} \tag{4.13}$$

where in (4.12) one takes either the upper or the lower signs. As clearly seen from (4.11), it is not a function of $x + y$ contrary to the standard binomial. This is why $(x;y)_q^m$ is used instead of the notation $(x+y)_q^m$ that can be found in some previous articles (e.g. Ruegg 1990).

Lemma 4. Equations (4.1) and (4.2) have a simultaneous solution if and only if

$$k_1 = h_1 - h_2 \qquad k_2 = h_2. \tag{4.14}$$

This solution is unique up to a multiplicative factor and given by

$$P^{h_1 h_2}(\eta_i^s)|0\rangle = (\eta_1^1)^{h_1-h_2} \left(\eta_1^1 \eta_2^2; -q^{-(h_1-h_2+2)/4} \, \eta_2^1 \eta_1^2\right)_q^{h_2} |0\rangle. \tag{4.15}$$

Proof. By using the q-Bargmann representation (3.6) as well as properties (4.9) and (4.13), it is straightforward to show that when (4.14) is fulfilled, the q-boson state (4.15) satisfies both (4.1) and (4.2).

The second part of the proof is more involved. Let us first determine the extra conditions to be fulfilled by $A_{cd}^{(ab)}$ so that the q-boson states defined in (4.4) are also solutions of (4.2). Equation (4.2a) leads to the relations

$$A_{cd}^{(ab)} = 0 \qquad \text{if} \quad h_1 + h_2 - 2a - b \neq k_1 \quad \text{or} \quad a - b \neq k_2. \tag{4.16}$$

Hence the summations over a, b disappear as these parameters are given by

$$a = \tfrac{1}{3}(h_1 + h_2 - k_1 + k_2) \qquad b = \tfrac{1}{3}(h_1 + h_2 - k_1 - 2k_2). \tag{4.17}$$

Moreover, (4.2*b*) imposes that the following recursion relations be satisfied by $A_{cd}^{(ab)}$:

$$q^{(h_1+h_2-2a-b+2)/4} \, [c-d+1] \, A_{c+1,d}^{(ab)} + [a-c+d] \, A_{cd}^{(ab)} = 0 \tag{4.18}$$

$$q^{(a-b+2)/4} \, [d+1] \, A_{c,d+1}^{(ab)} + [b-d] \, A_{cd}^{(ab)} = 0 \tag{4.19}$$

where c and d vary in the ranges $a + b - h_2 - 1 \leqslant c \leqslant \min(h_1, a+b)$, $\max(0, c-a+1) \leqslant d \leqslant \min(b, c)$, and $a + b - h_2 \leqslant c \leqslant \min(h_1, a+b)$, $\max(0, c-a-1) \leqslant d \leqslant \min(b-1, c)$, respectively.

Considering successively (4.19) for $d = c-a-1, c-a, \ldots, \min(b-1, c)$, and any c in the range $a < c \leqslant \min(h_1, a+b)$ shows that the coefficients $A_{cd}^{(ab)}$ vanish whenever the conditions $0 \leqslant b \leqslant h_2$, $a + b - h_2 \leqslant c \leqslant a$, $0 \leqslant d \leqslant \min(b, c)$ are not fulfilled. Equation (4.18) for $c = a, a-1, \ldots, a+b-h_2$, and any d in the range $1 \leqslant d \leqslant \min(b, c)$, then indicates that the only surviving coefficients correspond to $0 \leqslant b \leqslant h_2$, $a + b - h_2 \leqslant c \leqslant a$, and $d = 0$. Considering (4.19) again for $0 < b \leqslant h_2$,

$a+b-h_2 \leqslant c \leqslant a$, and $d=0$, only leaves the coefficients $A^{(a0)}_{c0}$, which by iterating (4.18) can be expressed in terms of $A^{(a0)}_{a0}$ as follows:

$$A^{(a0)}_{c0} = (-1)^{a-c} q^{(a-c)(h_1+h_2-2a+2)/4} \begin{bmatrix} a \\ c \end{bmatrix} A^{(a0)}_{a0} \qquad 0 \leqslant a-h_2 \leqslant c \leqslant a. \tag{4.20}$$

It is easily checked that (4.18) and (4.19) do not impose any further restriction on $A^{(a0)}_{a0}$.

It now only remains to combine (4.20) with conditions (4.5), (4.6) and (4.17). For $b=d=0$, (4.5) becomes

$$A^{(a0)}_{c0} = (-1)^{h_2-a+c} q^{-(h_1-h_2+2)(h_2-a+c)/4} \begin{bmatrix} h_2 \\ a-c \end{bmatrix} A^{(a0)}_{a-h_2,0}. \tag{4.21}$$

Introducing (4.21) and a similar result for $A^{(a0)}_{a0}$ into (4.20) leads to

$$\left(q^{(a-c)(a-h_2)/2} \begin{bmatrix} h_2 \\ a-c \end{bmatrix} - \begin{bmatrix} a \\ a-c \end{bmatrix} \right) A^{(a0)}_{a-h_2,0} = 0 \qquad 0 \leqslant a-h_2 \leqslant c \leqslant a. \tag{4.22}$$

Hence $a=h_2$, and the only non-vanishing coefficients are

$$A^{(h_2 0)}_{c0} = (-1)^c q^{-(h_1-h_2+2)c/4} \begin{bmatrix} h_2 \\ c \end{bmatrix} A^{(h_2 0)}_{00} \qquad 0 \leqslant c \leqslant h_2. \tag{4.23}$$

Equation (4.6) is then automatically satisfied, while (4.17) directly leads to (4.14). By inserting (4.23) into (4.4) and taking (4.9) into account, one finally obtains the solution (4.15), whose uniqueness has thus been proved. □

Since (4.1) and (4.2) have a solution for any $h_1, h_2 \in \mathbb{N}$, such that $h_1 \geqslant h_2$, the results obtained in the present section can be summed up in the form of a theorem.

Theorem 2. For the q-algebras considered in theorem 1, the reduction of a $u_q(6)$ irrep $[h0\ldots0]$ into a sum of $su_q(3)+u_q(2)$ irreps is given by

$$[h0\ldots0] \downarrow \sum_{\substack{h_1 \geqslant h_2 \\ h_1+h_2=h}} \left((h_1-h_2, h_2) + [h_1 h_2] \right) \tag{4.24}$$

where there is no multiplicity.

5. Casimir operators of $su_q(3)$ and $u_q(2)$

As a consequence of the complementarity between su(3) and u(2) within single-row irreps of u(6), the boson realizations of their Casimir operators are functionally dependent on one another. In the present section we show that this property can be extended to the q-boson realizations of the corresponding q-algebras.

The Casimir operators of q-algebras, in particular those of $su_q(n)$, have been the subject of various recent studies (Pasquier and Saleur 1990, Chakrabarti 1991, Rodríguez-Plaza 1991, Zhang *et al* 1991, Gould *et al* 1991, Bincer 1991). Here we shall follow the approach of Rodríguez-Plaza (1991), based upon a theorem of Faddeev *et al* (1988). In the latter, the generators $Z_k, k = 0, 1, \ldots, n-1$, of the centre of $\mathrm{su}_q(n)$ are constructed in terms of the universal R-matrix and the fundamental representation of this q-algebra. The $n-1$ independent Casimir operators of $\mathrm{su}_q(n)$ are then defined as those linear combinations of the Z_k's that approach the Casimir operators of $\mathrm{su}(n)$ in the limit $q \to 1$.

In the notations used in the present paper, the two generators of the centre of $\mathrm{su}_q(2)$ can be written as

$$\mathcal{Z}_0 = \left(q^{1/2} - q^{-1/2}\right)^{-2}\left(q^{1/2} + q^{-1/2}\right) I \tag{5.1a}$$

$$\mathcal{Z}_1 = \mathcal{E}^{12}\mathcal{E}^{21} + \left[\tfrac{1}{2}(\mathcal{E}^{11} - \mathcal{E}^{22})\right] \left[\tfrac{1}{2}(\mathcal{E}^{11} - \mathcal{E}^{22}) - 1\right] + \left(q^{1/2} - q^{-1/2}\right)^{-2}\left(q^{1/2} + q^{-1/2}\right) I . \tag{5.1b}$$

From them, we find the single Casimir operator

$$\begin{aligned} \mathcal{C}_{2q} = \mathcal{Z}_1 - \mathcal{Z}_0 &= \mathcal{E}^{12}\mathcal{E}^{21} + \left[\tfrac{1}{2}\left(\mathcal{E}^{11} - \mathcal{E}^{22}\right)\right] \left[\tfrac{1}{2}\left(\mathcal{E}^{11} - \mathcal{E}^{22}\right) - 1\right] \\ &= \mathcal{E}^{21}\mathcal{E}^{12} + \left[\tfrac{1}{2}\left(\mathcal{E}^{11} - \mathcal{E}^{22}\right)\right] \left[\tfrac{1}{2}\left(\mathcal{E}^{11} - \mathcal{E}^{22}\right) + 1\right] \end{aligned} \tag{5.2}$$

whose limit when $q \to 1$ is the usual su(2) quadratic Casimir operator

$$\mathcal{C}_2 = \tfrac{1}{2} \sum_{st} \hat{\mathcal{E}}^{st} \hat{\mathcal{E}}^{ts} \tag{5.3}$$

where

$$\hat{\mathcal{E}}^{st} = \mathcal{E}^{st} - \tfrac{1}{2}\delta^{st} \sum_{u} \mathcal{E}^{uu} . \tag{5.4}$$

Since we deal here with $\mathrm{u}_q(2)$ instead of $\mathrm{su}_q(2)$, we have to supplement $\mathcal{C}_{2q}$ with the q-analogue of the u(2) linear Casimir operator $\mathcal{C}_1 = \sum_s \mathcal{E}^{ss}$, namely

$$\mathcal{C}_{1q} = \left[\mathcal{E}^{11} + \mathcal{E}^{22}\right] . \tag{5.5}$$

In the $\mathrm{su}_q(3)$ case, the centre of the q-algebra is generated by the operators

$$Z_0 = \left(q^{1/2} - q^{-1/2}\right)^{-2}\left(q + 1 + q^{-1}\right) I \tag{5.6a}$$

$$\begin{aligned} Z_1 = {} & q^{(H_1+2H_2-3)/6} E_{12}E_{21} + q^{(H_1-H_2+3)/6} E_{13}^{(-)} E_{31}^{(-)} \\ & + q^{-(2H_1+H_2-3)/6} E_{23}E_{32} + \left(q^{1/2} - q^{-1/2}\right)^{-2} \\ & \times \left(q^{(2H_1+H_2-3)/3} + q^{-(H_1-H_2)/3} + q^{-(H_1+2H_2-3)/3}\right) \end{aligned} \tag{5.6b}$$

$$\begin{aligned} Z_2 = {} & q^{-(H_1+2H_2-3)/6} E_{12}E_{21} + q^{-(H_1-H_2+3)/6} E_{13}^{(+)} E_{31}^{(+)} \\ & + q^{(2H_1+H_2-3)/6} E_{23}E_{32} + \left(q^{1/2} - q^{-1/2}\right)^{-2} \\ & \times \left(q^{-(2H_1+H_2-3)/3} + q^{(H_1-H_2)/3} + q^{(H_1+2H_2-3)/3}\right) \end{aligned} \tag{5.6c}$$

where use is made of the operators $E_{13}^{(\pm)}$, $E_{31}^{(\pm)}$, defined in (2.12). From them, two Casimir operators can be constructed, namely

$$\begin{aligned}
C_{2q} &= Z_1 + Z_2 - 2Z_0 \\
&= \tfrac{1}{2}\left(q^{(H_1+2H_2-3)/6} + q^{-(H_1+2H_2-3)/6}\right) E_{12}E_{21} + \tfrac{1}{2}\left(q^{(H_1+2H_2+3)/6}\right. \\
&\quad \left.+ q^{-(H_1+2H_2+3)/6}\right) E_{21}E_{12} + \tfrac{1}{2}\left(q^{(2H_1+H_2-3)/6} + q^{-(2H_1+H_2-3)/6}\right) E_{23}E_{32} \\
&\quad + \tfrac{1}{2}\left(q^{(2H_1+H_2+3)/6} + q^{-(2H_1+H_2+3)/6}\right) E_{32}E_{23} + \tfrac{1}{2} q^{-(H_1-H_2+3)/6} \\
&\quad \times \left(E_{13}^{(+)}E_{31}^{(+)} + E_{31}^{(+)}E_{13}^{(+)}\right) + \tfrac{1}{2} q^{(H_1-H_2+3)/6}\left(E_{13}^{(-)}E_{31}^{(-)} + E_{31}^{(-)}E_{13}^{(-)}\right) \\
&\quad + \tfrac{1}{2}(q+q^{-1})\left(\left[\tfrac{1}{3}(2H_1+H_2)\right]^2 + \left[\tfrac{1}{3}(H_1+2H_2)\right]^2\right) + \left[\tfrac{1}{3}(H_1-H_2)\right]^2 \\
&= \left(q^{(H_1+2H_2+3)/6} + q^{-(H_1+2H_2+3)/6}\right) E_{21}E_{12} + \left(q^{(2H_1+H_2+3)/6}\right. \\
&\quad \left.+ q^{-(2H_1+H_2+3)/6}\right) E_{32}E_{23} + q^{-(H_1-H_2+3)/6} E_{31}^{(+)}E_{13}^{(+)} \\
&\quad + q^{(H_1-H_2+3)/6} E_{31}^{(-)}E_{13}^{(-)} + \left[\tfrac{1}{3}(2H_1+H_2+3)\right]^2 \\
&\quad + \left[\tfrac{1}{3}(H_1+2H_2+3)\right]^2 + \left[\tfrac{1}{3}(H_1-H_2)\right]^2 - 2
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
C_{3q} &= 2(q^{1/2} - q^{-1/2})^{-1}(Z_1 - Z_2) \\
&= \left[\tfrac{1}{3}(H_1+2H_2-3)\right] E_{12}E_{21} + \left[\tfrac{1}{3}(H_1+2H_2+3)\right] E_{21}E_{12} \\
&\quad - \left[\tfrac{1}{3}(2H_1+H_2-3)\right] E_{23}E_{32} - \left[\tfrac{1}{3}(2H_1+H_2+3)\right] E_{32}E_{23} \\
&\quad + (q^{1/2} - q^{-1/2})^{-1}\left[q^{(H_1-H_2+3)/6}\left(E_{13}^{(-)}E_{31}^{(-)} + E_{31}^{(-)}E_{13}^{(-)}\right)\right. \\
&\quad \left. - q^{-(H_1-H_2+3)/6}\left(E_{13}^{(+)}E_{31}^{(+)} + E_{31}^{(+)}E_{13}^{(+)}\right)\right] \\
&\quad + 2\left[\tfrac{1}{3}(2H_1+H_2)\right]\left[\tfrac{1}{3}(H_1+2H_2)\right]\left[\tfrac{1}{3}(H_1-H_2)\right] \\
&\quad + \left[\tfrac{1}{3}(4H_1+2H_2)\right] - \left[\tfrac{1}{3}(2H_1+4H_2)\right] \\
&= 2\left[\tfrac{1}{3}(H_1+2H_2+3)\right] E_{21}E_{12} - 2\left[\tfrac{1}{3}(2H_1+H_2+3)\right] E_{32}E_{23} \\
&\quad + 2(q^{1/2}-q^{-1/2})^{-1}\left(q^{(H_1-H_2+3)/6} E_{31}^{(-)}E_{13}^{(-)} - q^{-(H_1-H_2+3)/6} E_{31}^{(+)}E_{13}^{(+)}\right) \\
&\quad + 2\left[\tfrac{1}{3}(H_1-H_2)\right]\left[\tfrac{1}{3}(2H_1+H_2+3)\right]\left[\tfrac{1}{3}(H_1+2H_2+3)\right].
\end{aligned} \tag{5.8}$$

In the limit $q \to 1$, they go to

$$C_2 = \sum_{ij} \hat{E}_{ij}\hat{E}_{ji} \qquad C_3 = \tfrac{2}{3}\sum_{ijk} \hat{E}_{ij}\hat{E}_{jk}\hat{E}_{ki} - \sum_{ij} \hat{E}_{ij}\hat{E}_{ji} \tag{5.9}$$

respectively, where

$$\hat{E}_{ij} = E_{ij} - \tfrac{1}{3}\delta_{ij}\sum_k E_{kk}. \tag{5.10}$$

It is now easy to establish the following result:

Theorem 3. When the $su_q(3)$ and $u_q(2)$ generators are realized by (3.12) and (3.14) respectively, the corresponding Casimir operators satisfy the equations

$$C_{2q} = \left(q^{(N+3)/6} + q^{-(N+3)/6}\right)\mathcal{C}_{2q} + \left[\tfrac{1}{3}(N+3)\right]^2 + \left[\tfrac{1}{6}(N+6)\right]^2 + \left[\tfrac{1}{6}N\right]^2 - 2 \tag{5.11a}$$

$$C_{3q} = 2\left[\tfrac{1}{3}(N+3)\right]\left\{\mathcal{C}_{2q} - \left[\tfrac{1}{6}(N+6)\right]\left[\tfrac{1}{6}N\right]\right\} \tag{5.11b}$$

where N is related to $\mathcal{C}_{1q}$ by

$$\mathcal{C}_{1q} = [N]. \tag{5.12}$$

Proof. Direct verification using the commutation relations (3.1). □

Remark. Theorem 3 can be checked by comparing the eigenvalues of the $su_q(3)$ Casimir operators corresponding to an irrep $(h_1 - h_2, h_2) = [h_1 h_2 0]$

$$\langle C_{2q}\rangle = \left[\tfrac{1}{3}(2h_1 - h_2 + 3)\right]^2 + \left[\tfrac{1}{3}(h_1 + h_2 + 3)\right]^2 + \left[\tfrac{1}{3}(h_1 - 2h_2)\right]^2 - 2 \tag{5.13a}$$

$$\langle C_{3q}\rangle = 2\left[\tfrac{1}{3}(2h_1 - h_2 + 3)\right]\left[\tfrac{1}{3}(h_1 + h_2 + 3)\right]\left[\tfrac{1}{3}(h_1 - 2h_2)\right] \tag{5.13b}$$

with those of the $u_q(2)$ Casimir operators associated with the complementary irrep $[h_1 h_2]$

$$\langle \mathcal{C}_{1q}\rangle = [h_1 + h_2] \tag{5.14a}$$

$$\langle \mathcal{C}_{2q}\rangle = \left[\tfrac{1}{2}(h_1 - h_2)\right]\left[\tfrac{1}{2}(h_1 - h_2 + 2)\right]. \tag{5.14b}$$

6. q-boson realization of the $su_q(3)$ Gel'fand–Tseitlin basis states

The purpose of the present section is to build the GT orthonormal basis for an $su_q(3)$ irrep $(h_1 - h_2, h_2)$, in the realization where its highest-weight state is the q-boson state (4.15). The basis vectors

$$\left|\begin{matrix} h_1 & & h_2 & & 0 \\ & p_1 & & p_2 & \\ & & r & & \end{matrix}\right\rangle \qquad h_1 \geqslant p_1 \geqslant h_2 \geqslant p_2 \geqslant 0 \qquad p_1 \geqslant r \geqslant p_2 \tag{6.1}$$

are specified by definite irreps $[p_1 p_2]$ and $[r]$ of the $u_q(2)$ and $u_q(1)$ q-subalgebras of $su_q(3)$, spanned by I, $H_1 + H_2$, H_2, E_{12}, E_{21}, and I, $H_1 + H_2$, respectively. With respect to the $u_q(2)$ q-algebra complementary to $su_q(3)$, all of them are highest-weight states of irreps characterized by the same Young diagram $[h_1 h_2]$.

According to Ueno *et al* (1989), the GT vectors (6.1) can be constructed in two steps from the normalized highest-weight state

$$|h_1 h_2\rangle \equiv \left|\begin{matrix} h_1 & & h_2 & & 0 \\ & h_1 & & h_2 & \\ & & h_1 & & \end{matrix}\right\rangle \tag{6.2}$$

by using lowering operators for $u_q(3)$ and $u_q(2)$, whose explicit form is given by (Quesne 1992)

$$L_3^1 = E_{31}[E_{11} - E_{22} + 1] + E_{21}E_{32}q^{-(E_{11}-E_{22}+1)/2} \tag{6.3a}$$

$$L_3^2 = E_{32} \tag{6.3b}$$

and

$$L_2^1 = E_{21} \tag{6.4}$$

respectively. First, one determines from $|h_1 h_2\rangle$ the semi-maximal states, i.e. those states of highest weight in the $u_q(2)$ q-subalgebra of $su_q(3)$,

$$\left|\begin{matrix} h_1 & h_2 & 0 \\ & p_1 & p_2 \end{matrix}\right\rangle \equiv \left|\begin{matrix} h_1 & & h_2 & & 0 \\ & p_1 & & p_2 & \\ & & p_1 & & \end{matrix}\right\rangle = N_{p_1 p_2}^{h_1 h_2 0}(L_3^2)^{h_2 - p_2}(L_3^1)^{h_1 - p_1}|h_1 h_2\rangle \tag{6.5}$$

where the normalization factor is given by (Ueno *et al* 1989, Quesne 1992)

$$N_{p_1 p_2}^{h_1 h_2 0} = \left(\frac{[p_1 - h_2]!\,[p_1 + 1]!\,[p_2]!\,[p_1 - p_2 + 1]!}{[h_1 - p_1]!\,[h_1 - p_2 + 1]!\,[h_2 - p_2]!\,[h_1 - h_2]!\,[h_1 + 1]!\,[h_2]!}\right)^{1/2}. \tag{6.6}$$

Second, one obtains the general basis states (6.1) from the semi-maximal ones by using the relation

$$\left|\begin{matrix} h_1 & & h_2 & & 0 \\ & p_1 & & p_2 & \\ & & r & & \end{matrix}\right\rangle = N_r^{p_1 p_2}(L_2^1)^{p_1 - r}\left|\begin{matrix} h_1 & h_2 & 0 \\ & p_1 & p_2 \end{matrix}\right\rangle \tag{6.7}$$

where

$$N_r^{p_1 p_2} = \left(\frac{[r - p_2]!}{[p_1 - p_2]!\,[p_1 - r]!}\right)^{1/2}. \tag{6.8}$$

To apply this procedure, we need the explicit form of the normalized highest-weight state (6.2). It is given in the following lemma:

Lemma 5. The q-boson state

$$|h_1 h_2\rangle = A^{h_1 h_2} P^{h_1 h_2}(\eta_i^s)|0\rangle \tag{6.9}$$

where $P^{h_1 h_2}(\eta_i^s)$ is defined in (4.15), and $A^{h_1 h_2}$ by

$$A^{h_1 h_2} = q^{h_2/4} B^{h_1 h_2} \qquad B^{h_1 h_2} = \left(\frac{[h_1 - h_2 + 1]}{[h_1 + 1]!\,[h_2]!}\right)^{1/2} \tag{6.10}$$

is the normalized highest-weight state of an $su_q(3)$ irrep $(h_1 - h_2, h_2)$.

Proof. In the corresponding boson case, the normalization coefficient $A^{h_1h_2}$ is calculated by making use of some properties that have to do with the differentiation of a determinant, more specifically with the effect of the Cayley operator upon the mth power of its determinant (Turnbull 1960). In the case of a 2 × 2 determinant, the latter is given by

$$\left(\partial_1^1\partial_2^2 - \partial_2^1\partial_1^2\right)\left(z_1^1 z_2^2 - z_2^1 z_1^2\right)^m = m(m+1)\left(z_1^1 z_2^2 - z_2^1 z_1^2\right)^{m-1}. \quad (6.11)$$

The q-analogue of this relation results from (4.12) and (4.13) and can be written as

$$\left(D_1^1 D_2^2 - q^{(\alpha-m-1)/2} D_2^1 D_1^2\right)\left(z_1^1 z_2^2; -q^{-\alpha/2} z_2^1 z_1^2\right)_q^m$$
$$= q^{-1/2}[m]\,[m+1]\left(z_1^1 z_2^2; -q^{-(\alpha+1)/2} z_2^1 z_1^2\right)_q^{m-1} \qquad \alpha \in \mathbb{R}. \quad (6.12)$$

The q-Bargmann representation, as well as (4.12) and (6.12), lead to the recursion relation

$$\left(A^{h_1h_2}\right)^2 = q^{1/2}([h_1+1][h_2])^{-1}\left(A^{h_1-1,h_2-1}\right)^2 \quad (6.13)$$

from which $\left(A^{h_1h_2}\right)^2$ can be expressed in terms of $\left(A^{h_1-h_2,0}\right)^2$, given by (3.3). The final result is (6.10). □

It is now straightforward to obtain the explicit form of the GT vectors.

Lemma 6. The semi-maximal and general GT basis states of the $\mathrm{su}_q(3)$ irrep (h_1-h_2, h_2), whose highest-weight state is given in (6.9), can be written as

$$\left|\begin{matrix} h_1 & & h_2 & & 0 \\ & p_1 & & p_2 & \end{matrix}\right\rangle = A_{p_1p_2}^{h_1h_2} P_{p_1p_2}^{h_1h_2}(\eta_i^s)|0\rangle \quad (6.14)$$

$$\left|\begin{matrix} h_1 & & h_2 & & 0 \\ & p_1 & & p_2 & \\ & & r & & \end{matrix}\right\rangle = A_{p_1p_2r}^{h_1h_2} P_{p_1p_2r}^{h_1h_2}(\eta_i^s)|0\rangle \quad (6.15)$$

where

$$P_{p_1p_2}^{h_1h_2}(\eta_i^s) = (\eta_1^1)^{p_1-h_2}(\eta_3^1)^{h_1-p_1}\left(\eta_1^1\eta_2^2; -q^{-(p_1-p_2+2)/4}\,\eta_2^1\eta_1^2\right)_q^{p_2}$$
$$\times\left(\eta_1^1\eta_3^2; -q^{-(h_1-h_2-p_2+2)/4}\,\eta_3^1\eta_1^2\right)_q^{h_2-p_2} \quad (6.16)$$

$$P_{p_1p_2r}^{h_1h_2}(\eta_i^s) = \left(\eta_1^1\right)^{r-h_2}\sum_\mu\left(-q^{-(h_1-h_2+r+2)/4}\right)^\mu \begin{bmatrix} p_1-p_2-\mu \\ r-p_2 \end{bmatrix}\begin{bmatrix} h_2-p_2 \\ \mu \end{bmatrix}$$
$$\times\left(\eta_2^1\right)^{p_1-r-\mu}\left(\eta_3^1\right)^{h_1-p_1+\mu}\left(\eta_1^1\eta_2^2; -q^{-(p_1-p_2-2\mu+2)/4}\,\eta_2^1\eta_1^2\right)_q^{p_2+\mu}$$
$$\times\left(\eta_1^1\eta_3^2; -q^{-(h_1-h_2+p_1-p_2-r-2\mu+2)/4}\,\eta_3^1\eta_1^2\right)_q^{h_2-p_2-\mu} \quad (6.17)$$

and

$$A^{h_1h_2}_{p_1p_2} = q^{(h_2+(h_1-p_1)(h_2-p_2))/4} B^{h_1h_2}_{p_1p_2} \tag{6.18a}$$

$$B^{h_1h_2}_{p_1p_2} = \left(\frac{[h_1-h_2+1]!\,[p_1-p_2+1]!}{[h_1-p_1]!\,[h_1-p_2+1]!\,[p_1-h_2]!\,[h_2-p_2]!\,[p_1+1]!\,[p_2]!} \right)^{1/2} \tag{6.18b}$$

$$A^{h_1h_2}_{p_1p_2r} = q^{(h_2+(h_1-p_1)(h_2-p_2)+p_2(p_1-r))/4} B^{h_1h_2}_{p_1p_2r} \tag{6.19a}$$

$$B^{h_1h_2}_{p_1p_2r} = \left(\frac{[p_1-p_2+1]\,[h_1-h_2+1]!\,[p_1-r]!\,[r-p_2]!}{[h_1-p_1]!\,[h_1-p_2+1]!\,[p_1-h_2]!\,[h_2-p_2]!\,[p_1+1]!\,[p_2]!} \right)^{1/2} . \tag{6.19b}$$

Proof. The demonstration is based upon (6.3)–(6.10) as well as the relations

$$\left(L^1_3\right)^{h_1-p_1} P^{h_1h_2}(\eta^s_i)|0\rangle = \frac{[h_1-h_2]!\,[h_1+1]!}{[p_1-h_2]!\,[p_1+1]!} P^{h_1h_2}_{p_1h_2}(\eta^s_i)|0\rangle \tag{6.20}$$

$$\left(L^2_3\right)^{h_2-p_2} P^{h_1h_2}_{p_1h_2}(\eta^s_i)|0\rangle = q^{(h_1-p_1)(h_2-p_2)/4} \frac{[h_2]!}{[p_2]!} P^{h_1h_2}_{p_1p_2}(\eta^s_i)|0\rangle \tag{6.21}$$

and

$$\begin{aligned}
&\left(L^1_2\right)^{p_1-r} P^{h_1h_2}_{p_1p_2}(\eta^s_i)|0\rangle \\
&\quad = q^{(p_1-r)(2h_2-p_2)/4} [p_1-r]! \left(\eta^1_3\right)^{h_1-p_1} \left(\eta^1_1\eta^2_2; -q^{-(p_1-p_2+2)/4}\,\eta^1_2\eta^2_1\right)^{p_2}_q \\
&\quad \times \sum_m q^{-m(p_1-p_2)/2} \begin{bmatrix} h_2-p_2 \\ m \end{bmatrix} \begin{bmatrix} p_1-h_2 \\ p_1-r-m \end{bmatrix} \left(\eta^1_1\right)^{r-h_2+m} \left(\eta^1_2\right)^{p_1-r-m} \\
&\quad \times \left(\eta^1_1\eta^2_3; -q^{-(h_1-h_2+p_1-p_2-r-2m+2)/4}\,\eta^1_3\eta^2_1\right)^{h_2-p_2-m}_q \\
&\quad \times \left(\eta^1_2\eta^2_3; -q^{-(h_1+h_2-r-2m+2)/4}\,\eta^1_3\eta^2_2\right)^{m}_q |0\rangle
\end{aligned} \tag{6.22}$$

which can be proved by induction over $h_1 - p_1$, $h_2 - p_2$, and $p_1 - r$, respectively. In the final step of the demonstration of (6.17) and (6.19), use is made of the identity

$$\sum_{k=0}^{m} (-1)^k q^{\pm(m-1)k/2} \begin{bmatrix} m \\ k \end{bmatrix} = \delta_{m,0} \tag{6.23}$$

leading to the relation

$$\begin{aligned}
\left(z^1_1\right)^m \left(z^1_2 z^2_3; -q^{\alpha} z^1_3 z^2_2\right)^m_q &= \sum_{\mu=0}^{m} (-q^{\alpha})^{\mu} \begin{bmatrix} m \\ \mu \end{bmatrix} \left(z^1_2\right)^{m-\mu} \left(z^1_3\right)^{\mu} \\
&\quad \times \left(z^1_1 z^2_3; -q^{\beta\pm\mu/2} z^1_3 z^2_1\right)^{m-\mu}_q \left(z^1_1 z^2_2; -q^{\beta-\alpha\pm(\mu-1)/2} z^1_2 z^2_1\right)^{\mu}_q
\end{aligned} \tag{6.24}$$

valid for any $m \in \mathbb{N}$, α, $\beta \in \mathbb{R}$, and either the upper or the lower sign choice. □

Remarks. (1) In (6.17) and (6.22), the range of the summation indices μ and m is restricted by the definition (4.3) of q-binomial coefficients, i.e. $0 \leqslant \mu \leqslant \min(h_2 - p_2, p_1 - r)$ and $\max(0, h_2 - r) \leqslant m \leqslant \min(h_2 - p_2, p_1 - r)$. (2) In the $q \to 1$ limit, the results contained in (6.15), (6.17), and (6.19), go over, as they must, into the corresponding su(3) results given, for instance, in equations (A2) and (A3) of Moshinsky and Chacón (1968).

The GT vectors given in lemma 6 are written in terms of monomials in η_i^1, and of q-binomials depending on tensor products of two q-boson creation operators and on the deformation parameter q raised to a power that varies with the state (and the term) considered. It is useful to express the vectors in an alternative form wherein the building blocks are independent of the state considered and, at the same time, the q-binomials are replaced by ordinary binomials.

This is possible by introducing an explicit dependence upon the number operators N_i^s, as shown in the following lemma:

Lemma 7. The relation

$$\left(q^{(N_i^1+N_i^2+1)/4}\eta_1^1\eta_i^2 - q^{-(N_i^1+N_i^2+1)/4}\,\eta_i^1\eta_1^2\right)^m |0\rangle = q^{m/4}\left(\eta_1^1\eta_i^2; -q^{-1/2}\,\eta_i^1\eta_1^2\right)_q^m |0\rangle \tag{6.25}$$

is valid for $i = 2$, 3 and any $m \in \mathbb{N}$.

Proof. Equation (6.25) can be demonstrated by induction over m starting from $m = 1$ and using (4.12). □

With the help of this lemma, it is easy to arrive at the searched-for expressions of the GT vectors, given in the following theorem:

Theorem 4. The maximal, semi-maximal, and general Gel'fand–Tseitlin basis states corresponding to an $\mathrm{su}_q(3)$ irrep $(h_1 - h_2, h_2)$, and given in lemmas 5 and 6, can be rewritten as

$$|h_1 h_2\rangle = B^{h_1 h_2}\left(\eta_{12}^{12}\right)^{h_2}\left(\eta_1^1\right)^{h_1-h_2}|0\rangle \tag{6.26}$$

$$\left|\begin{matrix} h_1 & h_2 & 0 \\ p_1 & p_2 & \end{matrix}\right\rangle = B_{p_1 p_2}^{h_1 h_2}\left(\eta_{12}^{12}\right)^{p_2}\left(\eta_{13}^{12}\right)^{h_2-p_2}\left(\eta_1^1\right)^{p_1-h_2}\left(\eta_3^1\right)^{h_1-p_1}|0\rangle \tag{6.27}$$

$$\left|\begin{matrix} h_1 & h_2 & 0 \\ p_1 & p_2 & \\ r & & \end{matrix}\right\rangle = B_{p_1 p_2 r}^{h_1 h_2}\sum_\mu\left(-q^{-(p_1-p_2-\mu+1)/2}\right)^\mu \begin{bmatrix} p_1 - p_2 - \mu \\ r - p_2 \end{bmatrix}\begin{bmatrix} h_2 - p_2 \\ \mu \end{bmatrix}$$
$$\times\left(\eta_{12}^{12}\right)^{p_2+\mu}\left(\overline{\eta}_{13}^{12}\right)^{h_2-p_2-\mu}\left(\eta_1^1\right)^{r-h_2}\left(\eta_2^1\right)^{p_1-r-\mu}\left(\eta_3^1\right)^{h_1-p_1+\mu}|0\rangle \tag{6.28}$$

where

$$\eta_{1i}^{12} = q^{(N_i^1+N_i^2+1)/4}\,\eta_1^1\eta_i^2 - q^{-(N_i^1+N_i^2+1)/4}\,\eta_i^1\eta_1^2 \qquad i = 1, 2 \tag{6.29}$$

$$\overline{\eta}_{13}^{12} = q^{(N_3^1+N_1^2+1)/4}\,\eta_1^1\eta_3^2 - q^{-(N_1^1+2N_2^1+N_3^2+1)/4}\,\eta_3^1\eta_1^2 \tag{6.30}$$

and the order of the non-mutually commuting operators on the right-hand side of the relations does matter.

Proof. The q-binomial depending upon $\eta^1_1\eta^2_2$ and $\eta^1_2\eta^2_1$ is first commuted to the right and replaced by a number-dependent ordinary binomial with the help of lemma 7. The latter binomial is then commuted to the left. This procedure leads to (6.26). In the cases of (6.27) and (6.28), it only remains to perform the same transformation on the second q-binomial depending upon $\eta^1_1\eta^2_3$ and $\eta^1_3\eta^2_1$. □

Remarks. (1) Although the operators η^s_i commute amongst themselves, they do not commute with η^{12}_{12}, η^{12}_{13}, $\bar{\eta}^{12}_{13}$, nor do the latter commute with one another. As already noted by Biedenharn and Lohe (1991a, b) in the $u_q(2)$ case, only monomials such as $q^{(N^1_i+N^2_1+1)/4}\eta^1_1\eta^2_i$ and $q^{-(N^1_1+N^2_i+1)/4}\eta^1_i\eta^2_1$ have simple commutation properties amongst themselves and with η^1_j. (2) The commutation properties of the operators (6.29) and (6.30) with the $su_q(3)$ and $u_q(2)$ generators are also quite complicated. Such intricacies are related to the problem of constructing irreducible tensors with respect to $su_q(3)$ (Rittenberg and Scheunert 1992), to which we hope to come back in a forthcoming publication.

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